Elliptic quintics on cubic fourfolds, O'Grady 10 and Lagrangian fibrations

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Joint work with Chunyi Li and Xiaolei Zhao (arXiv:2007.14108)

HK manifolds

Hyperkähler manifold: compact complex simply connected Kähler manifold X with $H^{2,0}(X) = \mathbb{C}\eta$, where η is a symplectic form.
$\rightsquigarrow$ projective HK manifolds.

Examples

dim 2: K3 surfaces.
dim > 2: 4 deformation classes are known.

1. (Beauville) $\mathrm{Hilb}^n(S)$ where S is a K3 surface, $n \geq 2$;
2. (Beauville) $\mathrm{Kum}^n(A)$ where A is an abelian surface, $n \geq 2$;
3. (O'Grady) 10-dimensional example OG10;
4. (O'Grady) 6-dimensional example OG6.

Today: we focus on OG10.

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More on Examples (1) and (3)

Let S be a K3 surface.

1. (Mukai, Yoshioka) Moduli spaces of stable sheaves on S with primitive Mukai vector $v \quad \sim_{\text{def}} \mathrm{Hilb}^n(S)$.
3. (O'Grady, Lehn-Sorger) Symplectic resolutions of moduli spaces of semistable sheaves on S with Mukai vector $v = 2v_0$, $v_0^2 = 2 \quad \sim_{\text{def}}$ OG10.

Surprising: cubic fourfolds have many associated HK manifolds.
A cubic fourfold Y is a smooth cubic hypersurface in $\mathbb{P}^5$ over $\mathbb{C}$.

1. (Beauville-Donagi) Fano variety F_Y parametrizing lines in Y $\sim_{\text{def}} \mathrm{Hilb}^2(\mathrm{K3})$.
2. (Lehn-Lehn-Sorger-van Straten) HK eightfold M_Y constructed out of twisted cubic curves, for Y not containing a plane $\sim_{\text{def}} \mathrm{Hilb}^4(\mathrm{K3})$.
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Why HK manifolds from cubic fourfolds?

By the work of Kuznetsov there is a subcategory of K3 type in $\mathrm{D}^{\mathrm{b}}(Y) := \mathrm{D}^{\mathrm{b}}(\mathrm{Coh}(Y))$, denoted by $\mathcal{K}u(Y)$.

Our goal

1. Construct examples of projective hyperkähler manifolds of type OG10 as desingularizations of moduli spaces of semistable objects in $\mathcal{K}u(Y)$.
2. Relate them to the geometry of Y:
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Let S be a K3 surface with polarization H.

Denote by $\tilde{H}(S,\mathbb{Z}) = (H^*(S,\mathbb{Z}), \langle\,,\,\rangle)$ the Mukai lattice of S.

Take $v_0 \in \tilde{H}_{\mathrm{alg}}(S,\mathbb{Z})$ with $\langle v_0, v_0\rangle = 2$ and $v = 2v_0$.

$M_H(v)$ = moduli space of H-Gieseker semistable sheaves on S with Mukai vector v.

Let H be a v-generic polarization on $S \rightsquigarrow$ strictly semistable sheaves are S-equivalent to $F \oplus F'$ with F, F' stable sheaves with Mukai vector $v_0 \rightsquigarrow \mathrm{Sing}(M_H(v)) \cong \mathrm{Sym}^2(M_H(v_0))$.

Example OG10 (O'Grady)

$v_0 = v(\mathcal{I}_Z)$, where $\mathcal{I}_Z$ = ideal sheaf of 2 points in S,
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$M_H(v)$ has a symplectic resolution $\tilde{M}$, obtained by blowing up the singular locus with the reduced scheme structure, which is a projective HK 10-fold $\sim_{def}$ OG10.

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K3 category of a cubic fourfold

Proposition (Kuznetsov)

$\mathrm{D}^{\mathrm{b}}(Y) = \langle \mathcal{K}u(Y), \mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2) \rangle$ *where*
$\mathcal{K}u(Y) := \{E \in \mathrm{D}^{\mathrm{b}}(Y) : \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(Y)}(\mathcal{O}_Y(i), E) = 0, \forall\, i = 0, 1, 2\}.$

Properties

- $\mathcal{K}u(X)$ is of K3 type, e.g. the Serre functor of $\mathcal{K}u(Y)$ is [2].
- (Addington-Thomas) The Mukai lattice $\tilde{H}(\mathcal{K}u(Y), \mathbb{Z})$ of $\mathcal{K}u(Y)$ is the free abelian group $\{\kappa \in K(Y)_{\mathrm{top}} : \chi([\mathcal{O}_Y(i)], \kappa) = 0, \text{ for all } i = 0, 1, 2\}$ with intersection form $-\chi$ and induced weight-two Hodge structure $\tilde{H}^{2,0}(\mathcal{K}u(Y)) := H^{3,1}(Y), \quad \tilde{H}^{1,1}(\mathcal{K}u(Y)) := \oplus_p H^{p,p}(Y)$.
- $H^*_{\mathrm{alg}}(\mathcal{K}u(Y)) := \tilde{H}^{1,1}(\mathcal{K}u(Y)) \cap \tilde{H}(\mathcal{K}u(Y), \mathbb{Z})$, then $\langle \lambda_1, \lambda_2 \rangle \cong A_2 := \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \subset H^*_{\mathrm{alg}}(\mathcal{K}u(Y))$ and there is a Hodge isometry $\langle \lambda_1, \lambda_2 \rangle^{\perp} \cong H^4(Y, \mathbb{Z})_{\mathrm{prim}}$.

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K3 category of a cubic fourfold

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Stability conditions on $\mathcal{K}u(Y)$

Theorem (Bayer, Lahoz, Macrì, Nuer, Perry, Stellari)

1. $\mathrm{Stab}(\mathcal{K}u(Y)) \neq \emptyset$. *They describe a connected component* $\mathrm{Stab}^\dagger(\mathcal{K}u(Y))$ *of* $\mathrm{Stab}(\mathcal{K}u(Y))$.
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$$\Delta \cong M_\sigma(v_0) \subset M^{\mathrm{sing}} \cong \mathrm{Sym}^2(M_\sigma(v_0)) \subset M$$

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Special case for applications

$v_0 = \lambda_1 + \lambda_2,\ v = 2\lambda_1 + 2\lambda_2 \quad \rightsquigarrow \quad \sigma\colon \tilde{M} \to M := M_\sigma(v).$

Goal: understand the objects in M.

Why? [Druel, Beauville] Let X be a smooth cubic threefold.
$M_{\mathrm{inst},X}$ = moduli space of rank 2 instanton sheaves on X, i.e. semistable sheaves with Chern character $(2, 0, -2, 0)$.

Objects in $M_{\mathrm{inst},X}$ are in one of the following classes:

1. Given an elliptic quintic curve $\Gamma \subset X$ (l.c.i. quintic curve with trivial canonical bundle, $h^0(\mathcal{O}_\Gamma) = 1$ and $\langle \Gamma \rangle \cong \mathbb{P}^4$).
$\rightsquigarrow 0 \to \mathcal{O}_X(-1) \to F_\Gamma \to \mathcal{I}_{\Gamma/X}(1) \to 0$
F_Γ is a rank 2 stable vector bundle.
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$v_0 = \lambda_1 + \lambda_2,\ v = 2\lambda_1 + 2\lambda_2 \quad \rightsquigarrow \quad \sigma\colon \tilde{M} \to M := M_\sigma(v).$

Goal: understand the objects in M.

Why? [Druel, Beauville] Let X be a smooth cubic threefold.

$M_{\mathrm{inst},X}$ = moduli space of rank 2 instanton sheaves on X,

i.e. semistable sheaves with Chern character $(2, 0, -2, 0)$.

Objects in $M_{\mathrm{inst},X}$ are in one of the following classes:

1. Given an elliptic quintic curve $\Gamma \subset X$ (l.c.i. quintic curve with trivial canonical bundle, $h^0(\mathcal{O}_\Gamma) = 1$ and $\langle \Gamma \rangle \cong \mathbb{P}^4$).
   $\rightsquigarrow 0 \to \mathcal{O}_X(-1) \to F_\Gamma \to \mathcal{I}_{\Gamma/X}(1) \to 0$
   F_Γ is a rank 2 stable vector bundle.
2. Given a smooth conic $C \subset X$, let θ_C be the theta characteristic of C.
   $\rightsquigarrow 0 \to F_C \to \mathcal{O}_X \otimes H^0(X, \theta_C(1)) \to \theta_C(1) \to 0$
   F_C is a rank 2 torsion free stable sheaf.
3. $\mathcal{I}_{\ell_1/X} \oplus \mathcal{I}_{\ell_2/X}$ for two lines $\ell_1, \ell_2 \subset X$ is a semistable sheaf.

Moduli space of instanton sheaves

$J^2(X) =$ 1-cycles of degree 2 on X.
$M_{\text{inst},X} \xrightarrow{\mathfrak{c}_2} J^2(X), \;\; F \mapsto c_2(F)$.

Theorem (Druel, Markushevich-Tikhomirov, Beauville)

The moduli space $M_{\text{inst},X}$ is smooth and connected.
The morphism $\mathfrak{c}_2$ contracts the locus $\{F_C, C$ smooth conic$\}$ to $F^2 \cong F_X$, where F_X is the Fano surface of lines in X.
The morphism $\mathfrak{c}_2$ is isomorphic to the blow up $\mathrm{Bl}_{F^2}(J^2(X))$ of $J^2(X)$ along F^2.

Back to the cubic fourfold Y:

Remark

For a smooth hyperplane section $i \colon X \hookrightarrow Y$ and $F \in M_{\text{inst},X}$ we have

$$\mathrm{ch}(i_* F) = 2\lambda_1 + 2\lambda_2.$$

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Associated objects in $\mathcal{K}u(Y)$

$$\begin{tikzcd} \mathrm{D}^{\mathrm{b}}(Y) = \langle \mathcal{O}_Y(-2), \arrow[r, bend right, "\mathsf{R}_{\mathcal{O}_Y(-2)}"'] & \mathcal{O}_Y(-1), \arrow[r, bend right, "\mathsf{R}_{\mathcal{O}_Y(-1)}"'] & \mathcal{K}u(Y), & \mathcal{O}_Y \rangle \arrow[l, bend left, "\mathsf{L}_{\mathcal{O}_Y}"] \end{tikzcd}$$

Definition (Projection functor)

$\mathsf{pr}\colon \mathrm{D}^{\mathrm{b}}(Y) \to \mathcal{K}u(Y)$, $\mathsf{pr} = \mathsf{R}_{\mathcal{O}_Y(-1)}\mathsf{R}_{\mathcal{O}_Y(-2)}\mathsf{L}_{\mathcal{O}_Y}$.

Definition

Given an elliptic quintic curve $\Gamma \subset Y$, we define

$$E_\Gamma := \mathrm{pr}(\mathcal{I}_{\Gamma/Y}(1))$$

where $\mathcal{I}_{\Gamma/Y}$ is the ideal sheaf of Γ in Y.
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Moduli space $M := M_\sigma(2\lambda_1 + 2\lambda_2)$

Theorem (Li, P., Zhao)

1. *We have $E_\Gamma \cong i_* F_\Gamma$, where $i \colon X \hookrightarrow Y$ is a smooth hyperplane section.*
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Consequence: description of an open subvariety in the stable locus of the moduli space M.

Strictly semistable locus: Take $\sigma \in \mathrm{Stab}^\dagger(\mathcal{K}u(Y))$ v-generic. [Li,P.,Zhao] $P_\ell := \mathrm{pr}(\mathcal{I}_{\ell/X})$ is σ-stable, $v(P_\ell) = \lambda_1 + \lambda_2$. So

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We apply this result to study the relation of M with the (twisted) Intermediate Jacobian of Y.

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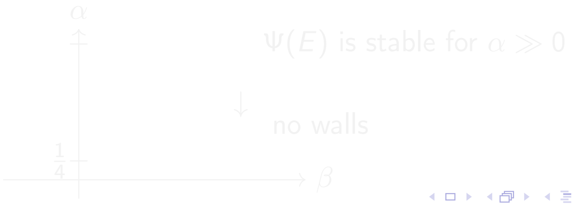
[Bayer-Lahoz-Macrì-Stellari]

$$\begin{tikzcd} & \mathrm{Bl}_\ell(Y) \arrow[dl] \arrow[dr, "\text{conic fibration}"] & \\ \ell \subset Y & & \mathbb{P}^3 \end{tikzcd}$$

$\mathcal{B}_0$ = even part of the sheaf of Clifford algebras associated to the conic fibration.

$\mathrm{D}^\mathrm{b}(\mathrm{Coh}(\mathbb{P}^3, \mathcal{B}_0)) =: \mathrm{D}^\mathrm{b}(\mathbb{P}^3, \mathcal{B}_0) = \langle \Psi(\mathcal{K}u(Y)), \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3 \rangle$.

$\sigma_{\alpha,-1}$ tilt-stability condition on $\mathrm{D}^\mathrm{b}(\mathbb{P}^3, \mathcal{B}_0) \rightsquigarrow \sigma := \sigma^0_{\alpha,-1}|_{\mathcal{K}u(Y)}$ for $\alpha < \frac{1}{4}$.



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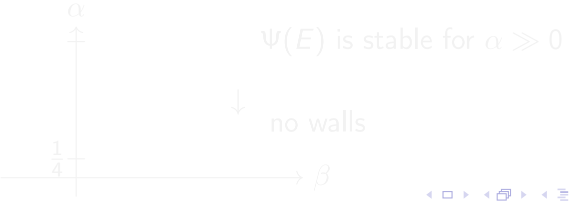

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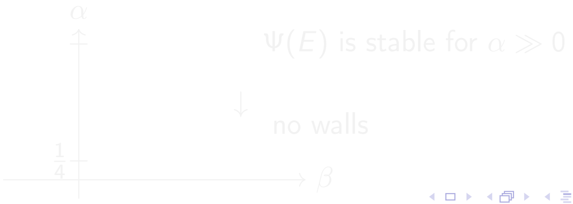

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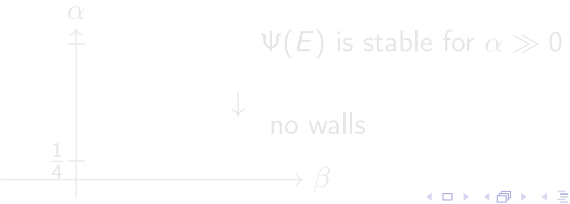

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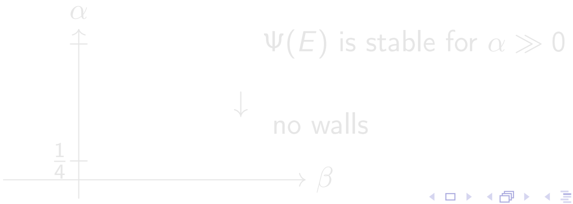

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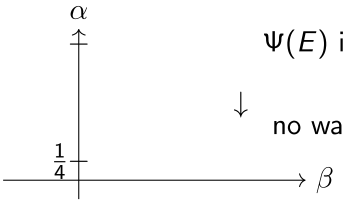

$\Psi(E)$ is stable for $\alpha \gg 0$

$\downarrow$ no walls

Intermediate Jacobian of Y

Consider the family of smooth hyperplane sections of Y

$$\mathcal{X} \to \mathbb{P}_0 \subset (\mathbb{P}^5)^\vee, \ \mathcal{X}_t \mapsto t \in \mathbb{P}_0$$

$\rightsquigarrow$ family of twisted intermediate Jacobians

$$p\colon J \to \mathbb{P}_0, \ J^1(\mathcal{X}_t) \mapsto t.$$

[Donagi-Markman] J has a symplectic form.

A long standing question was the existence of a HK compactification of J, i.e. of a HK $\bar{J}$ and a Lagrangian fibration $\pi\colon \bar{J} \to (\mathbb{P}^5)^\vee$ making the diagram

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Proved for very general Y by Laza-Saccà-Voisin for the untwisted family and by Voisin for the twisted family.
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Application 1

Recall $\sigma\colon \tilde{M} \to M$. Set
$M_0 := \{[E_\Gamma] \in M, \Gamma \subset X \subset Y \text{ elliptic quintic in smooth } X\} \subset \tilde{M}$.
We have the rational map defined by the support:
$\tilde{M} \dashrightarrow (\mathbb{P}^5)^\vee$ defined on $M_0 \to \mathbb{P}_0$ by $E_\Gamma \mapsto \mathrm{supp} E_\Gamma$.

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There exists a projective HK manifold N birational to $\tilde{M}$ with a Lagrangian fibration compactifying $p\colon J \to \mathbb{P}_0$, i.e.

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Idea of proof: Combination of results in birational geometry of HK manifolds [Matsushita].

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Recall $\sigma\colon \tilde{M} \to M$. Set
$M_0 := \{[E_\Gamma] \in M, \Gamma \subset X \subset Y$ elliptic quintic in smooth $X\} \subset \tilde{M}$.
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Flop between $\tilde{M}$ and N

$M_{\mathrm{inst},\mathbb{P}_0} \to \mathbb{P}_0$: relative moduli space of instanton sheaves.

$\mathrm{Bl}_{-F}(J) \xrightarrow{b} J \to \mathbb{P}_0$: blowup of J along the involution of the relative Fano surface of lines.

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Some remarks

For a very general cubic fourfold Y:
$\widetilde{M}$ and N are not isomorphic and N is isomorphic to Voisin's construction.
(The Picard rank of $\widetilde{M}$ and N is two $\Rightarrow$ there exists a unique HK compactification of the twisted family with a Lagrangian fibration structure.)

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Application 2

Let $\mathcal{C} \subset \mathrm{Hilb}^{5m}(Y)$ be the connected component of elliptic quintic curves in Y.

Conjecture (Castravet)

$\mathcal{C}$ has maximally rationally connected (MRC) quotient birational to J.

Theorem (Li, P., Zhao)

The projection $\mathrm{pr}\colon \mathrm{D}^{\mathrm{b}}(Y) \to \mathcal{K}u(Y)$ *induces a rational map*

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Next

Given a K3 surface S, take $v_0 \in \tilde{H}_{\mathrm{alg}}(S, \mathbb{Z})$ primitive and $v = mv_0$.

Theorem (Kaledin-Lehn-Sorger)

If either $m \geq 2$ and $\langle v_0, v_0 \rangle > 2$ or $m > 2$ and $\langle v_0, v_0 \rangle \geq 2$ and H is v-generic, then $M_H(v)$ does not admit a symplectic resolution.

Theorem (Arbarello-Saccà)

Case when H is not v_0-generic, they construct a symplectic resolution using quiver varieties.

Question: Do analogous statements hold for moduli spaces of semistable objects in Kuznetsov components?

Theorem (Chen, P., Zhao, in progress)

Let Y be a cubic fourfold and X a Gushel–Mukai fourfold. Then the Formality Conjecture holds for semistable objects in $\mathcal{K}u(Y)$ and $\mathcal{K}u(X)$.

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Thanks!